Aufgabe 1:

1.

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\operatorname{rot}\vec{r} = \begin{pmatrix} \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \end{pmatrix} = \vec{0}$$

$$\vec{\nabla}r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{\vec{r}}{r} = \vec{e}_r$$

$$\operatorname{grad}\frac{1}{r} = \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right) = -\frac{\vec{r}}{r^3} = -\frac{\vec{e}_r}{r^2}$$

$$\vec{\nabla}\times(\vec{\nabla}\Phi) = \begin{pmatrix} \frac{\partial}{\partial y}\frac{\partial\Phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\Phi}{\partial y} \\ \frac{\partial}{\partial z}\frac{\partial\Phi}{\partial x} - \frac{\partial}{\partial x}\frac{\partial\Phi}{\partial z} \\ \frac{\partial}{\partial x}\frac{\partial\Phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\Phi}{\partial x} \end{pmatrix} = \vec{0}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{U}) = \frac{\partial}{\partial x}\left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)$$

$$\frac{\partial^2 U_x}{\partial y\partial z} - \frac{\partial^2 U_x}{\partial z\partial y} + \frac{\partial^2 U_y}{\partial z\partial x} - \frac{\partial^2 U_y}{\partial x\partial z} + \frac{\partial^2 U_z}{\partial x\partial y} - \frac{\partial^2 U_z}{\partial y\partial x} = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{U}) = \vec{e}_i\epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l U_m = \vec{e}_i(\delta_{il}\delta_{jm}\partial_j\partial_l U_m - \delta_{im}\delta_{jl}\partial_j\partial_l U_m)$$

$$= \vec{e}_i(\partial_j\partial_i U_j - \partial_j\partial_j U_i) = \vec{\nabla}(\vec{\nabla}\cdot\vec{U}) - \Delta\vec{U}$$

$$\vec{\nabla}\times(\Phi\vec{U}) = \epsilon_{ijk}\partial_j\Phi U_k$$

$$\epsilon_{ijk}\Phi\partial_j U_k + \epsilon_{ijk}U_k\partial_j\Phi = \Phi(\vec{\nabla}\times\vec{U}) + (\vec{\nabla}\Phi)\times\vec{U}$$

2.

$$\operatorname{div}\vec{U} = \operatorname{div}\vec{V} = 0, \operatorname{rot}\vec{U} = \operatorname{rot}\vec{V} = 0$$

$$\operatorname{div}(\vec{U}\times\vec{V}) = \partial_i\epsilon_{ijk}U_jV_k = \epsilon_{ijk}(\partial_i U_j)V_k + \epsilon_{ijk}U_j(\partial_i V_k) = 0$$

$$\left(\operatorname{rot}(\vec{U}\times\vec{V})\right)_i = \epsilon_{ijk}\partial_j\epsilon_{klm}U_lV_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j U_l V_m = \partial_m U_i V_m - \partial_j U_j V_i$$

$$= V_m(\partial_m U_i) + 0 - U_j(\partial_j V_i) - 0 = \left((\vec{V}\cdot\vec{\nabla})\vec{U} - (\vec{U}\cdot\vec{\nabla})\vec{V}\right)_i$$

3.

$$\operatorname{div}\vec{A}_1 = 0 \quad , \quad \operatorname{rot}\vec{A}_1 = (0,0,2)$$

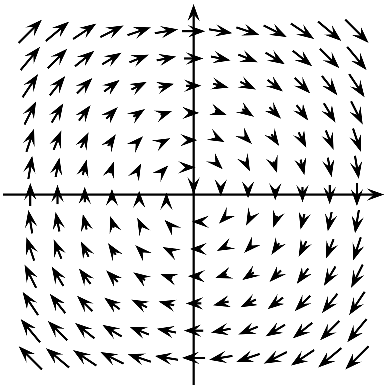
$$\operatorname{div}\vec{A}_2 = 0 \quad , \quad \operatorname{rot}\vec{A}_2 = (0,0,0)$$

$$\operatorname{div}\vec{A}_3 = \frac{\partial}{\partial x}\frac{x}{1+x^2} + \frac{\partial}{\partial y}\frac{y}{1+x^2} + \frac{\partial}{\partial z}\frac{z}{1+x^2} = \frac{(1+x^2) - 2x^2}{(1+x^2)^2} + \frac{2}{1+x^2}$$

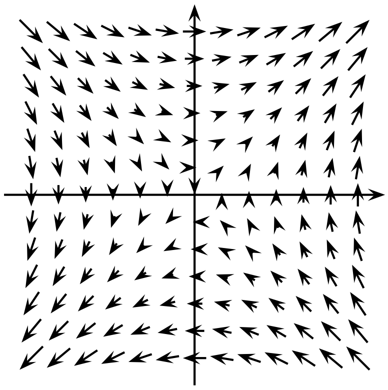
$$= \frac{1+x^2-2x^2+2+2x^2}{(1+x^2)^2} = \frac{3+x^2}{(1+x^2)^2}$$

$$\operatorname{rot}\vec{A}_3 = \operatorname{rot}\Phi\vec{r} = \Phi(\operatorname{rot}\vec{r}) + \operatorname{grad}\Phi\times\vec{r}$$

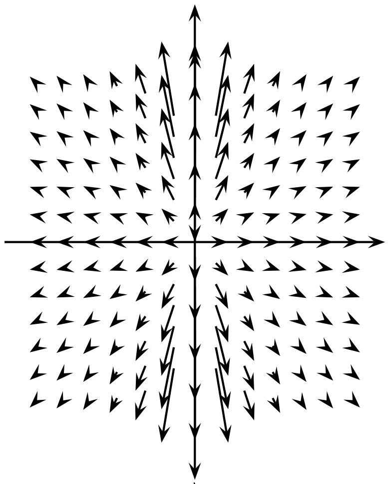
$$\operatorname{grad}\frac{1}{1+x^2}\times\vec{r} = \begin{pmatrix} \frac{-2x}{(1+x^2)^2} \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-2xz}{(1+x^2)^2} \\ \frac{-2xy}{(1+x^2)^2} \end{pmatrix}$$



$\vec{A}_1$



$\vec{A}_2$



$\vec{A}_3$

Aufgabe 2:

a)

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

Bew.:

$$y := ax$$

Betrag, weil sich im negativen Fall die Integrationsgrenzen umdrehen.

$$\int_{-\infty}^{\infty} f(x)\delta(ax) = \int_{-\infty}^{\infty} f(\frac{y}{a})\delta(y)\frac{1}{|a|}\mathrm{d}y$$

$$= \frac{f(0)}{|a|} = \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x$$

b)

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{\delta(x-x_i)}{|\frac{\mathrm{d}f}{\mathrm{d}x}(x_i)|} \quad \text{, mit } f(x) = 0\ \forall x = x_1, \ldots, x_N$$

Außerdem sei $f'(x) \neq 0\ \forall x = x_1, \ldots, x_N$.
Bew.: An den Nullstellen ist die Funktion lokal approximierbar mit $f(x+a) = f(x) + f'(x)a$.

c)

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}$$

$$= [f(x)\delta(x)] - \int f'(x)\delta(x)\mathrm{d}x = f'(0)$$

Aufgabe 3:

a)

$$\vec{A}(\vec{r}) = \vec{r} \Rightarrow \operatorname{div} A = 3$$

Wuerfel:

$$\int_{(F)} \vec{A}(\vec{r}) \cdot \mathrm{d}\vec{o} = 3abc$$

Kugel:

$$= 3\frac{4}{3}\pi r^3 = 4\pi r^3$$

Torus:

$$= 3\pi r^2 2R\pi = 6\pi^2 R r^2$$

b)

$$\vec{r}(r,\varphi) = (r\cos\varphi, r\sin\varphi) \qquad 0 \le r \le R$$

$$\operatorname{rot}\vec{A} = \begin{pmatrix} \partial_y 3xz - \partial_z 2yz \\ \partial_z x^2 y - \partial_x 3xz \\ \partial_x 2yz - \partial_y x^2 y \end{pmatrix} = \begin{pmatrix} -2y \\ -3z \\ -x^2 \end{pmatrix}$$

$$\int_{(F)} \begin{pmatrix} -2y \\ -3z \\ -x^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \int_{(F)} -x^2 \mathrm{d}o = \int_0^{2\pi}\int_0^R -r^2\cos^2\varphi r \mathrm{d}r\mathrm{d}\varphi = -\frac{R^4}{4}\int_0^{2\pi}\cos^2\varphi \mathrm{d}\varphi = -\frac{R^4}{4}\pi$$

$$\int \cos^2\varphi \mathrm{d}\varphi = \sin\varphi\cos\varphi + \int \sin^2\varphi \mathrm{d}\varphi$$

$$\Rightarrow 2\int \cos^2\varphi \mathrm{d}\varphi = \sin\varphi\cos\varphi + \varphi$$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}\varphi} = (-R\sin\varphi, R\cos\varphi, 0)$$

$$\vec{A}(\varphi) = (R^3\cos^2\varphi\sin\varphi, 0, 0)$$

$$\int_{\partial F} \vec{A}(\varphi)\mathrm{d}\vec{r} = \int_0^{2\pi} \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} R^3\cos^2\varphi\sin\varphi \\ 0 \\ 0 \end{pmatrix} \mathrm{d}\varphi = -R^4\int_0^{2\pi}\cos^2\varphi\sin^2\varphi \mathrm{d}\varphi = -R^4\frac{\pi}{4}$$