Aufgabe 1:

a)

$$\rho(\vec{r}) = \frac{3Q}{4\pi R^3}\Theta(R-r)$$
$$\vec{\omega} = \omega\vec{e}_z$$
$$\vec{r} = \sqrt{\frac{2\pi}{3}}r\begin{pmatrix}-(Y_{1,1}-Y_{1,-1})\\ i(Y_{1,1}+Y_{1,-1})\\ \sqrt{2}Y_{1,0}\end{pmatrix}$$
$$\vec{\jmath}(\vec{r}) = \rho(\vec{r})\vec{\omega}\times\vec{r} = -\frac{Q\omega r}{R^3}\sqrt{\frac{3}{8\pi}}\begin{pmatrix}-i(Y_{1,1}+Y_{1,-1})\\ (Y_{1,1}-Y_{1,-1})\\ 0\end{pmatrix}$$

b)

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \mathrm{d}^3r'\ \frac{\vec{\jmath}(\vec{r}')}{|\vec{r}-\vec{r}'|} \propto \int \mathrm{d}^3r'\ \vec{j}(\vec{r}')G(\vec{r}-\vec{r}') \quad , \quad G(r)\propto\frac{1}{r}$$
$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^{\,l}}{r^{l+1}}Y^*_{lm}(\Omega)Y_{lm}(\Omega') \qquad (r'<r)$$

Mitt der Orthonormalität der Kugelflächenfunktionen folgt:

$$\Rightarrow \vec{A}(\vec{r}) = \int \mathrm{d}^3r'\ \vec{\jmath}(\vec{r}')G(\vec{r}-\vec{r}') = \int \mathrm{d}^3r'\ \frac{\vec{j}(\vec{r})}{|\vec{r}-\vec{r}'|}$$
$$= -\frac{1}{c}\frac{Q\omega}{R^3}\sqrt{\frac{3}{8\pi}}\int_0^R \mathrm{d}r'\ r'^{\,2}\,r'\int \mathrm{d}\Omega'\ \frac{1}{|\vec{r}-\vec{r}'|}\begin{pmatrix}i(Y_{11}+Y_{1,-1})\\ Y_{11}-Y_{1,-1}\\ 0\end{pmatrix}$$
$$= -\frac{Q\omega}{cR^3}\sqrt{\frac{3}{8\pi}}\int_0^R \mathrm{d}r'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^{\,l+3}}{r^{l+1}}Y_{lm}\int \mathrm{d}\Omega'\ Y^*_{lm}(\Omega')\begin{pmatrix}i(Y_{11}+Y_{1,-1})\\ Y_{11}-Y_{1,-1}\\ 0\end{pmatrix}(\Omega')$$
$$= -\frac{Q\omega}{cR^3}\sqrt{\frac{3}{8\pi}}\int_0^R \mathrm{d}r'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^{\,l+3}}{r^{l+1}}Y_{lm}\begin{pmatrix}i\delta_{l1}\delta_{m1}+i\delta_{l1}\delta_{m,-1}\\ \delta_{l1}\delta_{m1}-\delta_{l1}\delta_{m,-1}\\ 0\end{pmatrix}$$
$$= -\frac{Q\omega}{cR^3}\sqrt{\frac{3}{8\pi}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{l+4}R^{l+4}\frac{4\pi}{2l+1}\frac{1}{r^{l+1}}Y_{lm}(\Omega)$$
$$= -\frac{Q\omega}{cR^3}\sqrt{\frac{3}{8\pi}}\begin{pmatrix}\frac{1}{5}R^5\frac{4\pi}{3}\frac{1}{r^2}\,i(Y_{11}+Y_{1,-1})\\ \frac{1}{5}R^5\frac{4\pi}{3}\frac{1}{r^2}\,(Y_{11}-Y_{1,-1})\\ 0\end{pmatrix}$$
$$= -\frac{Q\omega}{5c}\frac{R^2}{r^2}\begin{pmatrix}i(Y_{11}+Y_{1,-1})\\ Y_{11}-Y_{1,-1}\\ 0\end{pmatrix} = \frac{Q\omega}{5c}\frac{R^2}{r^3}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}$$

c)

$$\vec{A}(\vec{r}) = \frac{\vec{m} \times \vec{r}}{r^3} \quad , \quad \vec{m} = m\vec{e}_z$$

$$\Rightarrow m = \frac{Q\omega}{5c} R^2$$

$$\rho_{M,0} = \frac{3M}{4\pi R^3}$$

$$\vec{l} = \int \mathrm{d}^3 r \; \rho_M(\vec{r}) \vec{r} \times (\vec{\omega} \times \vec{r}) = \omega \int \mathrm{d}^3 r \; \rho_M(\vec{r}) \begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{2}{5} M \omega R^2 \vec{e}_z$$

$$m = g \frac{Q}{2Mc} \vec{l}$$

$$\Rightarrow g = 1$$

d)

$$\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r}) = \frac{Q\omega}{5c} \frac{R^2}{r^3}) ($$

Aufgabe 2:

$$v_E = \omega \cdot r_P \quad , \quad v_M = \omega \cdot r_M$$

$$\Rightarrow \frac{v_M}{v_E} = \frac{r_M}{r_P} \Rightarrow v_M > c$$

Das ist kein Widerspruch zur Relativitätstheorie, da sich weder Materie, noch Information schneller als das Licht ausbreitet. Betrachten wir das Problem nun infinitesimal unter Lorentz-Trafo:

$$\begin{pmatrix} \mathrm{d}t' \\ \mathrm{d}x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\mathrm{d}t \\ \mathrm{d}x \end{pmatrix}$$

$$\begin{pmatrix} c\mathrm{d}t \\ \mathrm{d}x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\mathrm{d}t' \\ 0 \end{pmatrix}$$

$$\mathrm{d}x = \gamma\beta c\mathrm{d}t \quad , \quad \mathrm{d}t = \gamma \mathrm{d}t'$$

$$\mathrm{d}x = v\gamma \mathrm{d}t'$$