Problems I

Due date: 8.11.05

Note: The problems can presumably not be solved in their entirety during the tutorials, if you did not prepare them at least partially in advance. I suggest you work over them as far as you can before the tutorial, so that we can concentrate on your problems, tricks, and in particular on the physics background. Quantum Field Theory is not learned by listening to a lecture.

Up-to-date information and pdf-files of the tutorials also at
http://theorie3.physik.uni-erlangen.de/lectures/ws2005_2006/griesshammer/QFTII.html.

1. The group SU(N) and its Lie algebra L[SU(N)]: All group elements of SU(N) can be written as $U = \exp \mathrm{i}\beta$.

   a) Show that β are Hermitean, traceless matrices. Count the linearly independent generators t^a. It is sufficient to consider the case $\beta \ll 1$ [why?].

   Hint: Once you have shown Hermiticity, you can simultaneously diagonalise U and β.

   b) Show that $(\partial_\mu U)U^\dagger = -U(\partial_\mu U^\dagger)$ and that it is anti-Hermitean.

   c) The structure constants of SU(N) are defined as

   $$\left[t^a, t^b\right] = \mathrm{i}\, f^{abc}\, t^c \ \ , \ \ \left\{t^a, t^b\right\} = \frac{1}{N}\, \delta^{ab} + d^{abc}\, t^c$$

   where f^{abc} is totally anti-symmetric, and d^{abc} totally symmetric. Show that

   $$\mathrm{tr}[t^a t^b t^c] = \frac{1}{4}\left(d^{abc} + \mathrm{i}\, f^{abc}\right) \ \ , \ \ \left[\sum_a t^a t^a, t^b\right] = 0 \ \ .$$

   d) What values has d^{abc} in SU(2)?

2. SU(2) and SO(3): As you should know, these two groups are closely related. We repeat only the well-known result from a new angle, to get a feeling about the relations between Lie groups and their algebrae. Consider the 3×3-matrix R with entries

   $$(R[U])^{ij} := \frac{1}{\sqrt{2}}\, \mathrm{tr}[\sigma^i\, U\, \sigma^j\, U^\dagger]$$

   which maps any element $U \in \mathrm{SU}(2)$ into some new group. We will now determine that group.

   a) Show that R is real and orthogonal, and that its determinant can only be 1, i.e. $R \in \mathrm{SO}(3)$.

   Hint: One possible way to get $\det R$ is to argue with U being continuous, and calculating just $\det R[U=1]$. Or you do it directly.

   b) By the mapping $R[U]$, the Lie algebra of SU(2) induces a Lie algebra on SO(3). As Lie algebras are locally unique, this *is* the Lie algebra of SO(3).
   Construct now the Lie algebra of the group formed by $R[U = 1 + \mathrm{i}t\beta^a \frac{\sigma^a}{2} + \mathcal{O}(t^2)]$, $t\beta^a \ll 1\ \ \forall a$.

   Result: The (un-normalised) orthogonal generators are the matrices $(\mathrm{i}\,\varepsilon^a)^{ij}$, where ε^{ijk} is the Levi-Civitá symbol, i.e. the totally anti-symmetric pseudo-tensor of rank 3, and $\epsilon^{ijk}\epsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$. What are the commutation relations?

   c) Take as generators of any Lie algebra the structure constants, written as matrices $(f^a)^{bc}$. This is called the "adjoint representation" of a Lie algebra. Show that it forms indeed a Lie algebra.

   d) Show that the mapping from U to R is 2-to-1. [You can show this also simultaneous with e).]

   e) Show that the group manifold of SU(2) is S^3, the 3-dimensional sphere, and determine the group manifold of SO(3).
   Hint: One possible way uses $(\sigma^i)^2 = 1$ (no sum over i), so that $\exp \mathrm{i}\beta^a\frac{\sigma^a}{2} = \cos\frac{\beta}{2} + \mathrm{i}\frac{\beta^a\sigma^a}{\beta}\,\sin\frac{\beta}{2}$, where $\beta = \sqrt{\beta^a\beta^a}$.