Problems IV

Due date: 10.1.06

Up-to-date information and pdf-files of the tutorials also at
http://theorie3.physik.uni-erlangen.de/lectures/ws2005_2006/griesshammer/QFTII.html.

1. ONE-LOOP RENORMALISATION OF QCD:

   a) Draw all lowest-order self-energy, vacuum-polarisation and vertex correction diagrams in QCD in a generic regularisation scheme. How divergent do you expect each diagram to be, and what combinations of colour-factors and momenta should it be proportional to?

   b) Which of the diagrams are zero in dimensional regularisation?

2. DIMENSIONAL REGULARISATION: Show by steps as in the lecture that in Minkowski-space

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \; \frac{q^\mu \, q^\nu}{[q^2 + 2q\cdot k - m^2 + \mathrm{i}\epsilon]^\alpha}$$
$$= \frac{\mathrm{i}}{(4\pi)^{d/2}\;\Gamma[\alpha]} \; \frac{(-1-\mathrm{i}\epsilon)^{d/2}}{(-k^2-m^2+\mathrm{i}\epsilon)^{\alpha-d/2}} \; \left[k^\mu \, k^\nu \; \Gamma[\alpha - \frac{d}{2}] + \frac{g^{\mu\nu}}{2}\left(-k^2 - m^2 + \mathrm{i}\epsilon\right) \; \Gamma[\alpha - 1 - \frac{d}{2}]\right] \; .$$

3. THRESHOLD EXPANSION AND DIMENSIONAL REGULARISATION. Let's have some fun with the fact that integrals without internal scales disappear in dimensional regularisation. The integral corresponding to a one-dimensional loop

$$I(a,b) := \int \mathrm{d}k \; \frac{1}{k^2 - a^2 + \mathrm{i}\epsilon} \; \frac{1}{k^2 - b^2 + \mathrm{i}\epsilon} = \frac{\mathrm{i}\pi}{ab(a+b)}$$

is usually solved by contour integration. Assume now $v^2 := \frac{a^2}{b^2} < 1$. By saddle-point approximation, the dominating contributions come from the regions where $|k|$ is close to a or b:

$$I(a,b) \approx \left[\int\limits_{|k|\sim a} + \int\limits_{|k|\sim b}\right] \mathrm{d}k \; \frac{1}{k^2 - a^2 + \mathrm{i}\epsilon} \; \frac{1}{k^2 - b^2 + \mathrm{i}\epsilon} \; .$$

In the first integral, $k \sim a$ is small against b, so that we can perform a Taylor expansion *to all orders* in $\frac{k}{b} \sim v < 1$. If $k^2 \gtrsim b^2$, the expansion breaks down, so that the approximated integrals can *not* be solved by contour integration. In general, the (arbitrary) borders of the integration régimes (the "cutoffs") lead to power-divergences. If one treats however

$$\frac{-1}{b^2} \sum_{n=0}^{\infty} \int\limits_{|k|\sim a} \mathrm{d}k \; \frac{1}{k^2 - a^2 + \mathrm{i}\epsilon} \; \frac{k^{2n}}{b^{2n}} \;\to\; \frac{-1}{b^2} \sum_{n=0}^{\infty} \int \mathrm{d}^d k \; \frac{1}{k^2 - a^2 + \mathrm{i}\epsilon} \; \frac{k^{2n}}{b^{2n}}$$

as a d-dimensional integral over all k with $d \to 1$ only at the end of the calculation, the contribution one obtains in the contour integration from the pole at $|k| = a$ emerges as a power series in $v = \frac{a}{b}$.

Convince yourself of this: First, extend the integration régime of the approximation around $k \sim a$ to the whole d-dimensional space. Then, calculate the integral order by order in the expansion, still treating $\frac{k^2}{b^2} \sim v^2$ as formally small. Make extensive use of the identities

$$k^{2n} = \sum_{m=0}^{n} \binom{n}{m} a^{2m} (k^2 - a^2)^{n-m} \quad \text{and} \quad \frac{k^2}{k^2 - a^2} = 1 + \frac{a^2}{k^2 - a^2}$$

The same can be done for the integration around b, $\frac{a}{k} \sim \frac{1}{v} > 1$. This is a pedgogical example of a very useful general formalism developed by Beneke and Smirnov: Nucl. Phys. **B522**, (1998) 321-344, taken from Grießhammer: Phys. Rev. **D58**, (1998) 094027 [Sorry about my self-centredness.].

A Merry Christmas and some quiet time between the years!