Problems VI

Due date: 7.2.06

Up-to-date information and pdf-files of the tutorials also at
http://theorie3.physik.uni-erlangen.de/lectures/ws2005_2006/griesshammer/QFTII.html.

Warning: This problem sheet is a lecture in disguise. We will treat here some details on the β-function of Yang-Mills Theories. It is therefore highly recommended that you attend this special tutorial on 7th February at 17:00h s.t. in SR 02.729 (Theory III) **even if you do usually not attend**. Next day's lecture will draw from the results of this tutorial.

1. THE RUNNING β-FUNCTION OF QCD: We construct the scale-dependence of the renormalised coupling constant $g_R(\mu) := g_R\ \mu^\epsilon$ from the renormalisation constants of non-Abelian gauge theories with gauge group $SU(N)$ and N_F quark flavours, in the MS-scheme of dimensional regularisation ($\epsilon = 2 - \frac{d}{2}$ in d space-time dimensions) and in the Lorentz gauge family with gauge parameter α to lowest non-trivial order in g_R. All relevant one-loop diagrams were discussed in tutorial IV.

   The quark self-energy diagram was calculated in the lecture for $\alpha = 1$, $m = 0$ and gives rise to the wave-function renormalisation constant Z_q for the quarks. The quark-loop contribution to the total gluonic vacuum-polarisation and wave-function renormalisation Z_A was the topic of the last tutorial. The quark-gluon vertex renormalisation constant Z_{qqA} is a longer story. One finds (see e.g. Ryder, Chap. 9.8):

$$Z_q = 1 - \frac{g_R^2}{(4\pi)^2}\ \alpha\ \frac{N^2-1}{2N}\ \frac{1}{\epsilon}\quad,\quad Z_A = 1 - \frac{g_R^2}{(4\pi)^2}\left[\frac{2N_F}{3} - \frac{N}{2}\ \left(\frac{13}{3} - \alpha\right)\right]\frac{1}{\epsilon}$$

$$Z_{qqA} = 1 - \frac{g_R^2}{(4\pi)^2}\left[\frac{3+\alpha}{4}\ N + \alpha\ \frac{N^2-1}{2N}\right]\frac{1}{\epsilon}$$

   a) Calculate the coupling-constant renormalisation Z_g of the bare coupling g to lowest order in the renormalised coupling g_R, cf. lecture, Sect. 1.d):

$$Z_g = \frac{g}{g_R\mu^\epsilon} = \frac{Z_{qqA}}{Z_q\ Z_A^{1/2}} = \frac{Z_{AAA}}{Z_A^{3/2}} = \frac{Z_{AAAA}^{1/2}}{Z_A} = \frac{Z_{\omega\bar{\omega}A}}{Z_\omega\ Z_A^{1/2}} = 1 - \frac{g_R^2}{(4\pi)^2}\ \frac{11N - 2N_F}{6}\ \frac{1}{\epsilon}$$

   b) Since the bare coupling g is independent of the subtraction constant μ, derive the dependence of the renormalised coupling $g_R(\mu) := g_R\ \mu^\epsilon$ on μ to lowest order, namely the function (result after limit $\epsilon \to 0$):

$$\beta(g_R(\mu)) = \mu\ \frac{\partial g_R(\mu)}{\partial\mu} = -\frac{11N - 2N_F}{3}\ \frac{g_R^3(\mu)}{(4\pi)^2}\quad .$$

   c) Solve the corresponding renormalisation group equation for the strong fine-structure constant $\alpha_s(\mu) = \frac{g_R^2(\mu)}{4\pi}$ at a given four-momentum squared Q^2, i.e. show with $t = \ln[Q^2/\mu^2]$ from

$$\frac{\partial\alpha_s(Q^2)}{\partial t} = -\frac{11N-2N_F}{3(4\pi)}\ \alpha_s^2(Q^2)\quad\text{that}\quad \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{4\pi}\ \frac{11N-2N_F}{3}\ \ln\frac{Q^2}{\mu^2}}\quad .$$

   d) Finally, we replace the μ-dependence by dimensional transmutation with

$$\Lambda_{\text{QCD}} = \mu\ \exp - \frac{2\pi}{\frac{11N-2N_F}{3}\alpha_s(\mu^2)}\quad\text{so that}\quad \alpha_s(Q^2) = \frac{4\pi}{\frac{11N-2N_F}{3}\ \ln\frac{Q^2}{\Lambda_{\text{QCD}^2}}}\quad .$$

   e) A host of experiments give $\alpha_s(Q^2 = M_Z^2) = 0.118 \pm 0.002$, $M_Z = 91.19$ GeV. Calculate Λ_{QCD} in the MS-scheme at this order. What should you pick for the number N_F of quark flavours? Reconcile your finding with the general notion that $\Lambda_{\text{QCD}} \approx 200$ MeV. Sketch $\alpha_s(Q^2)$ between e.g. 0.2 and 100 GeV. Watch out for the different number of active quarks as the scale changes. Does QCD predict its own breakdown? Compare to theories which are not asymptotically free, like QED.